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RELAXATION TECHNIQUE AS APPLIED TO WHEAT-STONE BRIDGE NETWORK PROBLEM

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ABSTRACT. This paper deals with the relaxational solution of the problem of Wheatstone bridge network. In doing so the heating effects of steady currents flowing in the network have been considered and thereafter the desired quantities are found out easily by relaxation method. It also shows how this method yields a number of useful informations at a time A comparison of the results obtained by this and other methods has been made.

INTRODUCTION

Bridges are one of the most widely used circuit arrangement in the field of measurement of circuit parameters, the simplest and important form of which is the Wheatstone bridge. Here a general case of unbalanced Wheatstone bridge (fig. 1), is considered in which the value of the galvanometer current has been found out. The normal methods of analysis of this bridge configuration consisting of three meshes is somewhat complicated for general values of bridge parameters. But here it will be shown that an easy and convenient solution of this problem is possible by relaxation method considering the equivalent circuit diagram (fig. 2) of the network shown in fig. 1

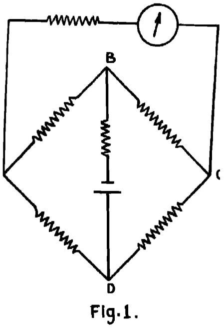

Fig. 1.

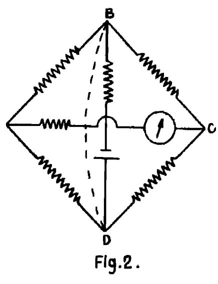

Fig. 2.

It was first of all shown by Southwell and Black (1938) and afterwards also by Dutta (1966), how relaxation method can be suitably applied to solve the problem of D.C. network. The underlying principle of this solution will reveal the utility of the technique in solving the network problem shown in fig. 2.

THE METHOD

The heating effects of steady currents flowing in the network have been considered in this method and an electrical theorem (Southwell and Black, 1938), in this connection has been used which can be stated as follows :

"In a network of conductors to which specified currents are supplied at two or more nodal points, the actual distribution of currents is such that the total generation of heat less twice the energy expended in supplying the specified currents from a source at datum potential has its minimum value consistent with the satisfaction of Kirchoff's second law".

If the two nodal points A and B of the Wheatstone bridge network shown in fig 2, be joined by a conductor of resistance R_{AB}, then by Ohm's law a current of $\frac{V_A - V_B}{R_{AB}}$ will flow from A to B, where V_A and V_B are the potentials at A and B.

Let the currents I_A and I_B flow towards A and B respectively and then,

$$-I_A = I_B = g_{AB}(V_A - V_B), \text{ where } g_{AB} = 1/R_{AB}.$$

Also if all the conductors connected to the nodal point A be considered it can be written that,

$$\Sigma\, g_{AB}(V_B - V_A) + I_{AO} = 0 \qquad \ldots \quad (1)$$

where I_{AO} means the current supplied to A from outside. Hence the heat generated in AB is given by the expression $g_{AB}(V_A - V_B)^2$, and thus the total heat generated in the network is represented by

$$2H = \sum_m g_{AB}(V_A - V_B)^2 \qquad \ldots \quad (2)$$

where $\sum\limits_m$ stands for the summation extending to every conductor. Let again a current be supplied to A from an outside source at datum potential V_0, so that the rate of expenditure of energy is measured by $I_{AO}(V_A - V_O)$ and the total expendidture of energy is measured by

$$\sum_n \{I_{AO}(V_A - V_O)\} = -E \qquad \ldots \quad (3)$$

where $\sum\limits_n$ means the summation extending to every nodal point. Thus equation (1) is typical of the conditions for a minimum value of the quantity,

$$Q = H + E = \tfrac{1}{2} \sum_m \{g_{AB}(V_A - V_B)^2\} + \sum_n \{I_{AO}(V_A - V_O)\} \qquad \ldots \quad (4)$$

as it is equivalent to,

$$-\frac{\partial Q}{\delta V_A} = -\frac{\delta}{\delta V_A}(H + E) = 0.$$

In this network problem as the source of e.m.f. (battery) is present it needs some modification in having its relaxational solution. In order to do this the whole e.m.f. of the source may be assumed to be utilized to pass the current to earth through the resistance of its own associated link and then a datum distribution in which the known currents enter and leave the network at the nodal point can be obtained Thereafter merely the effects of the neutralizing currents are to be calculated and superposed at those points.

Let the nodal points B and D be joined by a wire of zero resistance shown by dotted line in fig. 2. Now the current that will pass through the source from D to B will return by that wire and so in the datum distribution a current of E/R_{BD} enters the system at D and leaves at B, R_{BD} being the resistance in the branch BD. Next it is required to calculate and superpose the current distribution resulting when the neutralizing currents E/R_{BD} and $-E/R_{BD}$ are supplied to the net-work at B and D after removing the battery e.m.f.

If all the branches of the network shown in fig 2, be considered the expression for Q and the residuals can be written using the equation (4) as follows :

$$2Q = \frac{(V_A-V_B)^2}{R_{AB}} + \frac{(V_A-V_D)^2}{R_{AD}} + \frac{(V_B-V_D)^2}{R_{BD}} + \frac{(V_C-V_B)^2}{R_{CB}} + \frac{(V_C-V_D)^2}{R_{CD}}$$

$$+\frac{(V_C-V_A)^2}{R_{CA}} + 2\frac{E}{R_{BD}}\{V_O-V_B-(V_O-V_D)\} \qquad \dots \quad (5)$$

Hence,

$$\left.\begin{aligned}
-\frac{\delta Q}{\partial V_A} &= -\frac{(V_A-V_B)}{R_{AB}} - \frac{(V_A-V_D)}{R_{AD}} + \frac{(V_C-V_A)}{R_{CA}} = 0 = F_A \\
-\frac{\delta Q}{\delta V_B} &= \frac{(V_A-V_B)}{R_{AB}} - \frac{(V_B-V_D)}{R_{BD}} + \frac{(V_C-V_B)}{R_{CB}} + \frac{E}{R_{BD}} = \frac{E}{R_{BD}} = F_B \\
-\frac{\delta Q}{\delta V_C} &= -\frac{(V_C-V_B)}{R_{CB}} - \frac{(V_C-V_D)}{R_{CD}} - \frac{(V_C-V_A)}{R_{CA}} = 0 = F_C \\
-\frac{\delta Q}{\delta V_D} &= \frac{(V_A-V_D)}{R_{AD}} + \frac{(V_B-V_D)}{R_{BD}} + \frac{(V_C-V_D)}{R_{CD}} - \frac{E}{R_{BD}} = -\frac{E}{R_{BD}} = F_D
\end{aligned}\right\} \quad (6)$$

Liquidation of these residuals (Allen, 1954), obtained initially will yield the potentials at the nodal points A, B, C and D (fig. 2) simultaneously.

The method is illustrated by the following example worked out by Frank (1959), using a different method.

In the Wheatstone bridge network shown in Fig. 1, $E = 28$ volts, $R_{BD} = 100$ ohms, $R_{CA} = 200$ ohms, $R_{AB} = 200$ ohms, $R_{CB} = 300$ ohms, $R_{AD} = 100$ ohms and $R_{CD} = 150$ ohms, when it is balanced. It is required to find out the current flowing through the galvanometer if R_{CD} is changed by 4%, making the bridge unbalanced.

The value of R_{CD} is found out to be 156 ohms when the bridge is unbalanced.

Substituting the numerical values in the relations (5) and (6), it is obtained as follows :

$$2Q = \frac{(V_A-V_B)^2}{200}+\frac{(V_A-V_D)^2}{100}+\frac{(V_B-V_D)^2}{100}+\frac{(V_C-V_B)^2}{300}+\frac{(V_C-V_D)^2}{156}$$
$$+\frac{(V_C-V_A)^2}{200}+2\times 0.28(V_D-V_B) \quad \dots \quad (7)$$

$$\left.\begin{aligned}
-\frac{\partial Q}{\delta V_A} &= -\frac{(V_A-V_B)}{200}-\frac{(V_A-V_D)}{200}+\frac{(V_C-V_A)}{200}=0=F_A\\
-\frac{\delta Q}{\delta V_B} &= \frac{(V_A-V_B)}{200}-\frac{(V_B-V_D)}{100}+\frac{(V_C-V_B)}{300}+0.28=0.28=F_B\\
-\frac{\delta Q}{\delta V_C} &= -\frac{(V_C-V_B)}{300}-\frac{(V_C-V_D)}{156}-\frac{(V_C-V_A)}{200}=0=F_C\\
-\frac{\delta Q}{\delta V_D} &= \frac{(V_A-V_D)}{100}+\frac{(V_B-V_D)}{100}+\frac{(V_C-V_D)}{156}-0.28=-0.28=F_D
\end{aligned}\right\} \quad \dots \quad (8)$$

Now using the relations (7) and (8) the basic operation table (Table I) can be written from which the relaxation table (Table II) is easily obtained in order to liquidate the residuals F_A, F_B, F_C and F_D In Table II the initial values are multiplied by 10^2 so that on liquidation of these residuals the potentials at the nodal points A, B, C and D of the unbalanced bridge can be obtained nearest to their values on being divided by 10^2.

The galvanometer current is found out to be 0.428 mA by relaxation method. The corresponding value of the same is calculated to be 0.429 mA by a conventional method (Frank, 1959), which is in very good agreement with that obtained by the relaxation method.

DISCUSSION

In the Wheatstone bridge network considered here the resistances are included both in the galvanometer and battery circuits. This does not in any way bring about any complication in the relaxation method described here. But the conventional method of analysis becomes somewhat complicated and more particularly due to the presence of the resistance in the battery circuit, for which this resistance

TABLE I

Basic operation Table

Operation Step	$\delta_A V$	δV_B	δV_C	δV_D	δF_A	δF_B	δF_C	δF_D
(1)	312	—	—	—	−6.24	1.56	1.56	3.12
(2)	—	312	—	—	1.56	−5.72	1.04	3.12
(3)	—	—	312	—	1.56	1.04	−4.60	2.00
(4)	—	—	—	312	3.12	3.12	2.00	−8.24
(5)[(1)+(2)×4.0]	312	1248	—	—	0	−21.32	5.72	15.6
(6)[(2)+(3)× −1.0]	—	312	−312	—	0	−6.76	5.64	1.12
(7)[(5)+(6)× −3.1538]	312	264.0144	983.9856	—	0	0	−12.0676	12.0676

TABLE II

Relaxation Table

Liquidation Step	δV_A	δV_B	δV_C	δV_D	F_A	F_B	F_C	F_D
	INITIAL VALUES				0	28.0	0	−28.0
(a)[(5)×1.3133]	409.7496	1638.9984	0	0	0	0	7.5123	−7.5123
(b)[(7)×0.6225]	194.2200	164.3490	612.5310	0	0	0	0	0
	603.9696	1803.3474	612.5310	0	0	0	0	0

is ignored in many cases at the cost of bridge performance, whereas in the relaxation method the resistance in the battery circuit has got to be considered in order to calculate the current in the datum distribution. Also it yields the values of all the potentials at the nodal points simultaneously. Herein lies the advantage of the method described in this paper

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